# Control of a mechatronic system

Computer-based Exercises for ME-326

Fall 2025

## Introduction

The objective of these exercises is to illustrate different aspects of controller design on a laboratory setup using the Python Control Systems Library. The experimental setup under consideration is a rotary flexible joint produced by Quanser to demonstrate real-world control challenges encountered in some industrial large-gear robotic equipment. Five computer exercise modules are planned. In the first module a linearised model of the system is used to study the performance of the closed-loop system with a proportional controller. Some PID and cascade controllers are designed and validated in simulation during the second module. The loop shaping method is used to design a PID and a lead-lag compensator for the system in the third module. The state and output feedback controllers are designed and tested in simulation in the fourth module. The last module concerns the design of a digital RST controller for the system. The work is done by the groups of three students.

The completed version of Jupyter notebook should be submitted in moodle by the respective due dates of the modules. The notebooks will be evaluated and counted for 10 points in the final grade.

> **Important Note**
>
> Only the comments in `markdown` blocks and outputs of the cell will be considered for grading. Any comments inside the code blocks or variable which are not printed to output will be ignored. Use `View > Render Notebook with Voilà` menu to preview the rendered file which will be graded.

## System Description

The system consists of a base unit and a rotatory unit. The base unit consists of a DC motor that drives a small pinion gear through an internal gearbox. The pinion gear is fixed to a larger middle gear that rotates on the load shaft. The chassis of the rotatory unit is attached to the load gear of the gearbox and can rotate freely. The chassis of the rotatory unit is connected to a flexible link with some small weights attached to the link. (see Figure 1)

Two different states are considered:

- The rotation angle of the load shaft and the chassis of the motor: $\theta(t)$
- The deflection angle of the flexible link: $\alpha(t)$

In the following, we are interested in the angle between the link and the base unit: $\theta(t) + \alpha(t)$. Hence, the output of the system is taken to be from a speed encoder $y(t) = \theta(t) + \alpha(t)$. The input of the system $u(t)$ is the DC voltage applied to the DC motor.

### Modelling

The objective of this section is to find the transfer function between the system's output $y$ and the input $u$. A schematic diagram of the system is shown in Figure 2.


Figure 1: System under consideration: DC motor with a rotary flexible link

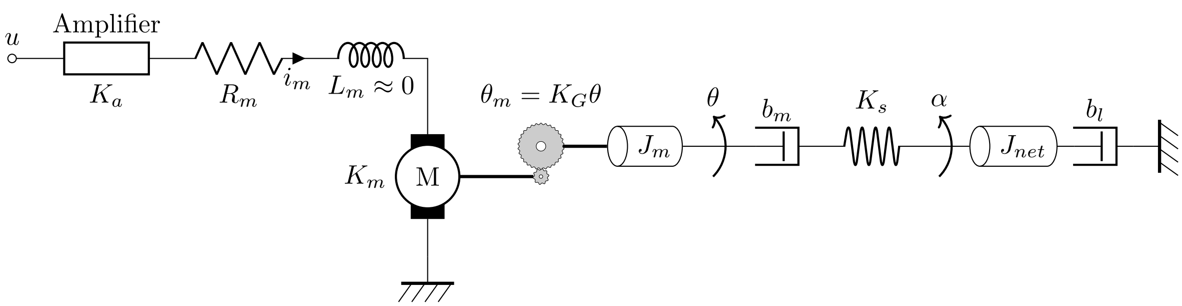


Figure 2: Schematic diagram of the whole system

First, the dynamic equations of the electrical and mechanical part of the DC motor are obtained. It is supposed that the motor's stator consists of permanent magnets that provide a constant magnetic field and the armature inductance can be neglected. Using Kirchhoff's voltage law, the following equation can be written for the electrical component of the motor:

$$K_a u(t) = R_m i_m(t) + E_{emf}(t) \tag{1}$$

where the electromotive force induced voltage is equal to $E_{emf}(t) = K_m \dot{\theta}_m(t)$. From the above equation, the armature current is obtained as:

$$i_m(t) = \frac{K_a u(t) - K_m \dot{\theta}_m(t)}{R_m} \tag{2}$$

By applying Newton's law to the motor shaft we get:

$$J_{mot}\ddot{\theta}_m(t) + \frac{T_L(t)}{K_G} = T_m(t) \tag{3}$$

where $J_{mot}$ is the motor inertia, $T_m(t) = K_m i_m(t)$ the motor torque and $T_L(t)/K_G$ the load torque seen through the gear (considering no loss in the gear).

The flexible link is attached from the middle to the load gear of the DC motor. At a given $\alpha$, the link is deformed and stores some potential energy which is dependent on the stiffness of the link (see Figure 3). This potential energy can be given as:

$$V(t) = \frac{1}{2}K_s\alpha^2(t) \tag{4}$$

where $K_s$ represents the equivalent stiffness of the flexible link.

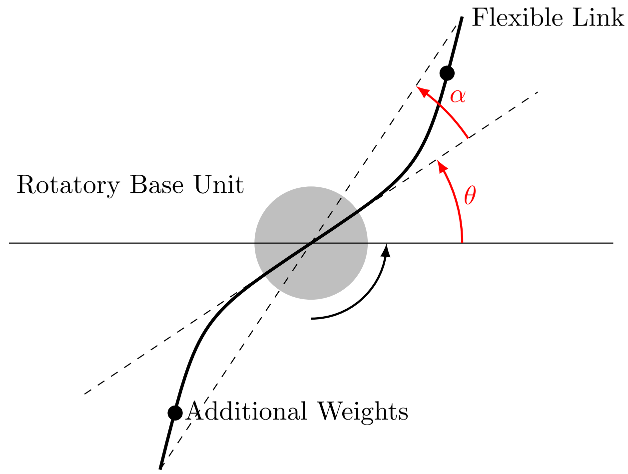


Figure 3: Schematic diagram of the rotatory flexible link

The kinetic energy of the flexible arm can be given by:

$$T(t) = \frac{1}{2}J_{mod}\theta^2(t) + \frac{1}{2}J_{net}\left(\dot{\theta}(t) + \dot{\alpha}(t)\right)^2 \tag{5}$$

where $J_{net}$ is the total moment of inertia of the flexible link and the attached weights on it. Assuming that the weights are symmetrically attached around the axis of rotation at a distance $l_w$:

$$J_{net} = \frac{1}{12}m_l l_l^2 + 2m_w l_w^2 \tag{6}$$

where, $m_l$ and $l_l$ are the mass and the length of the flexible link respectively, and $m_w$ is the mass of the attached weight.

The dynamics of the flexible arm can then be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \tag{7}$$

where, $q_i$ are the generalised coordinate and $Q_i$ are the generalised non-conservative force. Furthermore, the Lagrangian $L(t)$ is given as $T(t) - V(t)$.

For the flexible arm, take

$$q = \begin{bmatrix} \theta \\ \alpha \end{bmatrix} \quad \& \quad Q = \begin{bmatrix} T_L - b_{mod}\dot{\theta} \\ -b_l\dot{\alpha} \end{bmatrix}$$

Solving the Euler-Lagrange equation gives

$$J_{mod}\ddot{\theta}(t) + J_{net}(\ddot{\theta}(t) + \ddot{\alpha}(t)) + b_{mod}\dot{\theta}(t) = T_L(t) \tag{8}$$

$$J_{net}\ddot{\theta}(t) + J_{net}\ddot{\alpha}(t) + b_l\dot{\alpha}(t) + K_s\alpha(t) = 0 \tag{9}$$

Using the fact that $\theta_m = K_G\theta$, (3) can transformed using the Laplace transform:

$$K_G^2 J_{mot}s^2\theta(s) + T_L(s) = K_m K_G i_m(s) \tag{10}$$

and we replace

$$i_m(s) = \frac{K_a u(s) - K_m K_G s\theta(s)}{R_m}$$

which leads to

$$T_L(s) = \frac{K_m K_G}{R_m}\left(K_a u(s) - K_m K_G s\theta(s)\right) - K_G^2 J_{mot}s^2\theta(s) \tag{11}$$

Using the Laplace transform on (9) gives:

$$J_{net}s^2\theta(s) + J_{net}s^2\alpha(s) + b_l s\alpha(s) + K_s\alpha(s) = 0 \quad \Longrightarrow \quad \alpha(s) = -\frac{J_{net}s^2}{J_{net}s^2 + b_l s + K_s}\theta(s) \tag{12}$$

Using the Laplace transform on (8) gives:

$$J_{mod}s^2\theta(s) + J_{net}s^2(\theta(s) + \alpha(s)) + b_{mod}s\theta(s) = T_L(s) \tag{13}$$

and replacing $\alpha(s)$,

$$\left(J_{mod}s^2 + b_{mod}s + J_{net}s^2\frac{b_l s + K_s}{J_{net}s^2 + b_l s + K_s}\right)\theta(s) = T_L(s) \tag{14}$$

Using (11) replace $T_L(s)$ to obtain:

$$\left(J_{mod}s^2 + b_{mod}s + K_G^2 J_{mot}s^2 + J_{net}s^2\frac{b_l s + K_s}{J_{net}s^2 + b_l s + K_s} + K_m K_G s\frac{K_m K_G}{R_m}\right)\theta(s) = \frac{K_m K_G K_a}{R_m}u(s) \tag{15}$$

Let $J_m = J_{mod} + K_G^2 J_{mot}$, and simplify:

$$\begin{aligned}\left((J_{net}s^2 + b_l s + K_s)(R_m J_m s^2 + R_m b_{mod}s + K_m^2 K_G^2 s) + R_m J_{net}(b_l s + K_s)s^2\right)\theta(s)& \\ = (J_{net}s^2 + b_l s + K_s)K_m K_G K_a u(s)&\end{aligned} \tag{16}$$

Finally we obtain:

$$G_\theta(s) = \frac{(J_{net}s^2 + b_l s + K_s)K_m K_G K_a}{(J_{net}s^2 + b_l s + K_s)(R_m J_m s^2 + R_m b_{mod}s + K_m^2 K_G^2 s) + R_m J_{net}(b_l s + K_s)s^2} \tag{17}$$

$$G_\alpha(s) = \frac{-(J_{net}s^2)K_m K_G K_a}{(J_{net}s^2 + b_l s + K_s)(R_m J_m s^2 + R_m b_{mod}s + K_m^2 K_G^2 s) + R_m J_{net}(b_l s + K_s)s^2} \tag{18}$$

$$G(s) = \frac{(b_l s + K_s)K_m K_G K_a}{(J_{net}s^2 + b_l s + K_s)(R_m J_m s^2 + R_m b_{mod}s + K_m^2 K_G^2 s) + R_m J_{net}(b_l s + K_s)s^2} \tag{19}$$

## Numerical values

| | | | |
|---|---|---|---|
| Gain of power amplifier | $K_a$ | 1.0 | - |
| Motor resistance | $R_m$ | 2.2 | Ω |
| Torque constant | $K_m$ | 0.009 87 | N m/A |
| Gearbox ratio | $K_G$ | 60.0 | - |
| Motor Inertia | $J_{mot}$ | $3.87 \times 10^{-7}$ | $\mathrm{kg\,m^2}$ |
| Inertia of rotatory chassis | $J_{mod}$ | $3.922 \times 10^{-4}$ | $\mathrm{kg\,m^2}$ |
| Viscous damping | $b_{mod}$ | 0.005 | N m s/rad |
| Stiffness of flexible link | $K_s$ | 0.5 | N m/rad |
| Damping of flexible link | $b_l$ | 0.01 | N m s/rad |
| Mass of flexible link | $m_l$ | 0.2 | kg |
| Length of flexible link | $l_l$ | 0.5 | m |
| Mass of attached weights | $m_w$ | 0.025 | kg |
| Length of attached weights | $l_w$ | 0.15 | m |

The transfer function $G(s)$ is to be defined in Python. In all modules of the computer exercises, the model $G(s)$ will be used as the plant model which is being controlled.

## Steps for state-space model

First define the states of the system as $x = \begin{bmatrix}\dot{\theta} & \theta & \dot{\alpha} & \alpha\end{bmatrix}^T$.

Using the fact that $\theta_m = K_G\theta$, we can find $T_L$ from eqn. (2) and (3) as follows:

$$T_L = -K_G^2 J_{\text{mot}}\ddot{\theta} + K_G K_m \left(\frac{K_a u - K_m K_G \dot{\theta}}{R_m}\right)$$

Then, replace $T_L$ in eqn. (8) to find one of the state equations. The other state equation is eqn. (9).